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Global stability of systems with amplitude and rate saturation compensation

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SUMMARY

Rate and amplitude saturation of the actuator is common in practical control systems. When the actuator is rate and amplitude saturated, the control performance can deteriorate rapidly, and in the worse case, the closed-loop system can become unstable. It is therefore important that both types of saturations are properly compensated. Following the approach for compensating amplitude constraints, a scheme for compensating systems with both rate and amplitude saturation is proposed in this paper. The conditions for the compensated system to be globally stable are derived, and from this result, a procedure for designing the rate and amplitude saturation compensators is devised. As it is difficult to design both the rate and the amplitude saturation compensators simultaneously, a two-step approach is adopted. In the proposed compensator design procedure, the amplitude saturation compensator is designed first, followed by the rate saturation compensator. As the compensators designed using the proposed procedure satisfy the conditions for global stability, the compensated system is therefore globally stable. It is also shown that these compensators can be designed using the LMI technique. The implementation of the design procedure is demonstrated by a simulation example. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: global stability; amplitude and rate saturation; saturation compensation; strictly positive real control

1. INTRODUCTION

Rate and amplitude constraints are common in practical control systems. The amplitude constraint often arises from the physical size of the actuator, whilst the inertia of the driving mechanism in the actuator often leads to rate constraint. It is well known that if the actuator is amplitude saturated, the control performance can deteriorate rapidly, and the closed-loop system can even be unstable in the worst case. Clearly, the addition of rate saturation will in general make the situation worse, not better. It is, therefore, important to devise suitable compensation methods to ensure the closed-loop system remain stable when amplitude and/or rate saturation occur. It is shown in Reference [1] that a linear system subject to amplitude

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saturation of the actuator can be globally asymptotically stabilized, if and only if it is asymptotically null controllable with bounded controls. However, this result does not in general apply to linear feedback systems subject to amplitude constraint [2–4]. Several feedback control laws that ensure these nonlinear systems are globally asymptotically stable are proposed in References [5, 6]. In References [7–9], feedback control laws utilizing the low and high gain technique are derived, such that these nonlinear systems are semi-globally stable. Based on the singular perturbation approach, dynamical feedback control laws that tracks the set point subject to the maximum allowable control rate are proposed in Reference [10]. Similar to systems with only amplitude saturation, there are two approaches to compensate for systems subject to both amplitude and rate constraints. In the first approach, the controllers are designed taking into account both constraints [11–18]. A simple approach to tackle the saturation problem is to introduce a supervisory loop in the control system for adjusting the gain of the controller such that no rate and amplitude saturation of actuator occur [11–14]. A drawback of this approach is that the closed-loop response is likely to be sluggish, as the control tends to be very conservative. It is shown in Reference [15] that low gain feedback control laws can be designed, such that linear systems subject to both amplitude and rate saturation closed-loop system are semi-global stabilized. A continuous-time predictive control via state feedback for single-input single-output linear systems with actuator amplitude and rate saturation was derived in Reference [16], and the local stability of the closed-loop system is analysed using the Lyapunov function. By applying the separation principle and using a deadbeat observer for state estimation, it is shown in Reference [17] that global asymptotic stability and offset-free tracking of admissible set-point references are achieved for discrete-time systems without violating the constraints. An approach to design LQG-type of fixed-structure optimal control law that guarantees the domain of attraction for linear systems with independent amplitude and rate saturations is proposed in Reference [18]. In the second approach, the rate and the amplitude saturation compensators are designed separately [19–21] similar to that for compensating amplitude saturation [4], after the controller is designed assuming no saturation. The main advantage of this approach is that the closed-loop system can achieve the designed performance when there is no actuator saturation, as saturation compensators are activated only after saturation has occurred. Further, only linear control systems design techniques are required to design the controller and the saturation compensators [20]. The sequence of saturation in systems with both rate and amplitude constraints is important, as it leads to different analysis of the performance and stability of the closed-loop system. For physical systems, rate saturation is more likely to occur before amplitude saturation. This is because amplitude saturation only occurs after the actuator has reaches its limits. Before reaching this limit, the actuator is likely to be rate saturated first, especially when the change in control is large. Therefore, following the discussions in References [19–22], it is assumed here that rate constraint precedes amplitude constraint. By extending the results on the compensation for amplitude saturation, a compensation scheme for both rate and amplitude constraints is proposed in References [19–22]. Though some design guidelines for the rate and amplitude saturation compensators are proposed in these papers, only few stability results of the compensated system are presented there, and little results can be found elsewhere in the literature. For this reason, the main aim of this paper is to derive the global stability conditions for systems with both rate and amplitude compensation. The organization of the paper is as follows. In Section 2, the system with rate and amplitude constraints is presented. The model of the actuator with both constraints is then discussed, followed by the derivation of the compensators for these constraints. The conditions for global stability of the systems with rate and amplitude compensation are derived in Section 3. The design of both the rate and amplitude compensators that satisfy the global stability conditions are devised using the LMI in Section 4. The results presented in this paper are illustrated by a simulation example presented in Section 5.

2. COMPENSATION OF RATE AND AMPLITUDE CONSTRAINTS

Denote $R^{m\times n}$ the set of all real matrices with m rows by n columns, R^n the set of all real n dimension vector, and A' the transpose of matrix A.

Consider a linear multi-input multi-output (MIMO) system P described by the following minimal state-space realization.

$$\begin{aligned} \dot{x}_p(t) &= Ax_p(t) + Bu(t) \\ y(t) &= Cx_p(t) \end{aligned} \tag{1}$$

where $x_p(t) \in R^n$ is the system state, $u(t) \in R^m$ is the actuator output, $y(t) \in R^p$ is the system output. $A \in R^{n\times n}, B \in R^{n\times m}$ and $C \in R^{p\times n}$ are constant matrices. From (1), the transfer function matrix of the system P is $T_p(s) = C(sI - A)^{-1}B$.

Assume that the following linear controller K has been designed such that the closed-loop system without rate and amplitude saturation is asymptotically stable.

$$\begin{aligned} \dot{x}_k(t) &= Fx_k(t) + Ge(t) \\ v(t) &= Hx_k(t) \\ e(t) &= w(t) - y(t) \end{aligned} \tag{2}$$

where $w(t) \in R^P$ is the reference input, $v(t) \in R^m$ is the controller output, $x_k(t) \in R^q$ is the controller state. $F \in R^{q\times q}, G \in R^{q\times p}$ and $H \in R^{m\times q}$ are constant matrices, and the transfer function matrix of the controller is $K(s) = H(sI - F)^{-1}G$.

The rate and amplitude constraints for the actuator are given by

$$u(t) = \text{sat}[v(t)] = (\text{sat}[v_1(t)] \;\; \text{sat}[v_2(t)] \;\; \cdots \;\; \text{sat}[v_m(t)])'$$

and

$$\dot{u}(t) = \text{sat}[\dot{v}(t)] = (\text{sat}[\dot{v}_1(t)] \;\; \text{sat}[\dot{v}_2(t)] \;\; \cdots \;\; \text{sat}[\dot{v}_m(t)])'$$

where

$$\text{sat}[v_i(t)] = \begin{cases} u_i^+, & v_i > u_i^+ \\ v_i(t), & u_i^- \leqslant v_i \leqslant u_i^+ \\ u_i^- & v_i < u_i^- \end{cases}$$

and

$$\text{sat}[\dot{v}_i(t)] = \begin{cases} \dot{u}_i^+, & \dot{v}_i(t) > \dot{u}_i^+ \\ \dot{v}_i(t), & \dot{u}_i^- \leqslant \dot{v}_i(t) \leqslant \dot{u}_i^+ \\ \dot{u}_i^- & \dot{v}_i(t) < \dot{u}_i^- \end{cases}$$

The amplitude limits $u_i^+ > 0$, $u_i^- < 0$, and the rate limits $\dot{u}_i^+ > 0$, $\dot{u}_i^- < 0$ are assumed known.

In practical control systems, rate constraint often occurs before the amplitude constraint, because the inertia of the actuator limits the rate at which control can be implemented by the actuator. Therefore, it is assumed in this paper that the actuator with both rate and amplitude constraints can be represented by a model with the rate constraint followed by the amplitude constraint [19, 20], as shown in Figure 1. To model the rate constraint, it is necessary to introduce a differentiator first to obtain the rate of change in control, and an integrator afterwards to restore it back to amplitude. Since the output of this model ensures that the actuator is no longer subject to both rate and amplitude constraints, the actuator can therefore be omitted if this model is inserted in the closed-loop system.

For simplicity, it is proposed in References [19, 20] that the rate and the amplitude saturation are compensated separately, as shown in Figure 2. Similar to the compensation for amplitude saturation [4], the input of the rate and amplitude saturation compensators the discrepancy in either the rate of change in the control, or the amplitude of the control that cannot be implemented by the actuator. Denote the rate saturation compensator by P_r and the amplitude compensator by $P_a = P_{au} = P_{av}$.

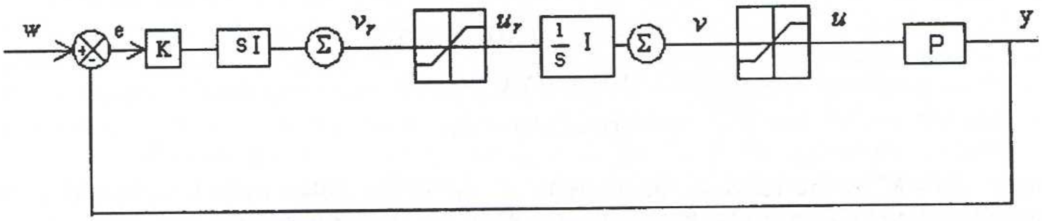

Figure 1. System with rate and amplitude saturation.

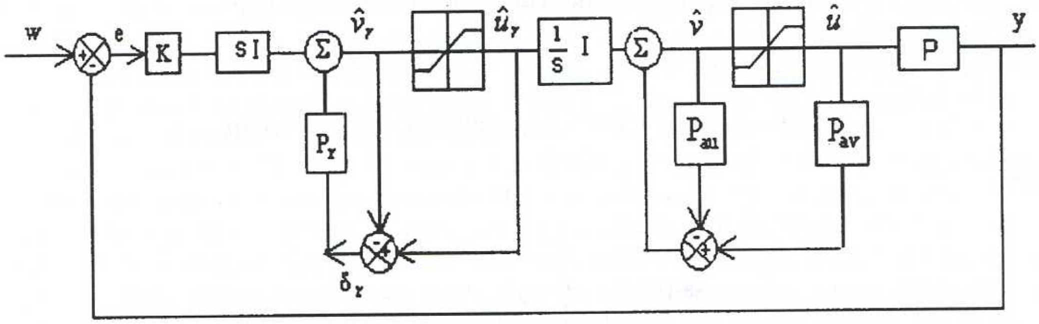

Figure 2. Rate and amplitude saturation compensated system.

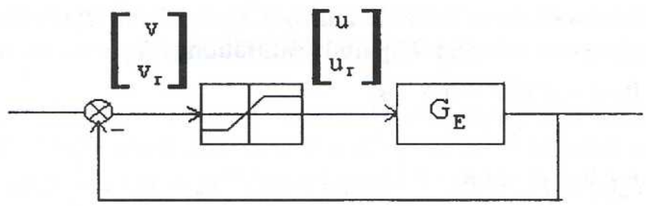

Figure 3. The equivalent system.

From (2), $\dot{v}(t) = HFx_k(t) + HGe(t)$ if there is no rate saturation. To compensate for the rate saturation, $\xi(t) \in R^m$ is added to $\dot{v}(t)$,

$$v_r(t) = HFx_k(t) + HGe(t) + \xi(t) \tag{3}$$

Where $\xi(t)$ is the output of the rate saturation compensator P_r given by

$$\begin{aligned} \dot{x}_\xi(t) &= A_\xi x_\xi(t) + B_\xi \delta_r(t) \\ \xi(t) &= C_\xi x_\xi(t) \end{aligned} \tag{4}$$

In (4), $\delta_r(t)$ is the rate of change in the control that cannot be implemented by the actuator,

$$\delta_r(t) = u_r(t) - v_r(t) \tag{5}$$

where

$$u_r(t) = sat(v_r(t))$$

and $x_\xi(t) \in R^s$. The matrices $A_\xi \in R^{s\times s}$, $B_\xi \in R^{s\times m}$ and $C_\xi \in R^{m\times s}$ in the compensator (4) are chosen by a user ensuring that the overall system remains globally stable when there is rate saturation.

Similarly, from Figure 1, $v(t)$ satisfies $\dot{v}(t) = u_r(t)$, when there is no amplitude saturation compensation. With amplitude saturation compensation, $\eta(t)$ is added to $\dot{v}(t)$ in the presence of amplitude saturation: $\dot{v}(t) = u_r(t) + \eta(t)$. Let $x_v(t)$ be a auxiliary variable, $v(t) = x_v(t)$, then

$$\begin{aligned} \dot{x}_v(t) &= u_r(t) + \eta(t) \\ v(t) &= x_v(t) \end{aligned} \tag{6}$$

The amplitude compensator is given by

$$\begin{aligned} \dot{x}_\eta(t) &= A_\eta x_\eta(t) + B_\eta \delta_a(t) \\ \eta(t) &= C_\eta x_\eta(t) + D_\eta \delta_a(t) \end{aligned} \tag{7}$$

$$\delta_a(t) = u(t) - v(t) \tag{8}$$

where $x_\eta(t) \in R^r$, $\eta(t) \in R^m$, and the matrices $A_\eta \in R^{r\times r}$, $B_\eta \in R^{r\times m}$, $C_\eta \in R^{m\times r}$ and $D_\eta \in R^{m\times m}$ are also design parameters that should be determined appropriately for the system to be globally stable under actuator saturation.

From (3)–(8), the transfer function matrices of (3) and (6) with rate and amplitude saturation compensation are

$$v_r(s) = qK(s)e(s) + P_r(s)\delta_r(s) \tag{9}$$

$$v(s) = \frac{1}{q}u_r(s) + P_{av}\hat{u}(s) - P_{au}\hat{v}(s) = \frac{1}{q}u_r(s) + P_a(s)\delta_a(s) \tag{10}$$

where s is the Laplace transform variable and $q = s$, $P_r(s)$ and $P_a(s)$ are, respectively, the transfer function matrices of rate and amplitude saturation compensators which is given by

$$P_r(s) = C_\xi(sI - A_\xi)^{-1}B_\xi \tag{11}$$

$$P_a(s) = P_{au} = P_{av} = \frac{1}{s}\left[C_\eta(sI - A_\eta)^{-1}B_\eta + D_\eta\right] = \frac{1}{s}P_\eta(s) \tag{12}$$

The reference input w is set to constant reference input in the following discussion of stability. To analyze the stability of the compensated system, it is rewritten as an equivalent system shown in Figure 3, which consists of a linear block and the nonlinear block representing actuator saturation. The compensated system has the same property as the following equivalent system.

From (1), (2), (4), (6) and (7):

$$G_E: \quad \begin{cases} \dot{x}_s(t) = A_s x_s(t) + B_s\bar{u}(t) \\ \bar{v}(t) = C_s x_s(t) \end{cases} \tag{13}$$

$$\bar{u}(t) = -\text{sat}(\bar{v}(t)) \tag{14}$$

where

$$A_s = \begin{bmatrix} A & 0 & 0 & 0 & 0 \\ -GC & F & 0 & 0 & 0 \\ 0 & 0 & -D_\eta & C_\eta & 0 \\ 0 & 0 & -B_\eta & A_\eta & 0 \\ B_\xi\text{HGC} & -B_\xi\text{HF} & 0 & 0 & A_\xi - B_\xi C_\xi \end{bmatrix}, \quad B_s = \begin{bmatrix} -B & 0 \\ 0 & 0 \\ -D_\eta & -I \\ -B_\eta & 0 \\ 0 & -B_\xi \end{bmatrix} \tag{15}$$

$$C_s = \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ -\text{HGC} & \text{HF} & 0 & 0 & C_\xi \end{bmatrix}, \quad \bar{u}(t) = -\begin{bmatrix} u(t) \\ u_r(t) \end{bmatrix}, \quad \bar{v}(t) = \begin{bmatrix} v(t) \\ v_r(t) \end{bmatrix} \tag{16}$$

and

$$x_s(t) = \left[x_p'(t)\ x_k'(t)\ x_v'(t)\ x_\eta'(t)\ x_\xi'(t)\right]' + \alpha$$

in which $\alpha = [0\ \ (F^{-1}Gw)'\ \ 0\ \ 0\ \ 0]'$. The transfer function matrix of the system (13) is

$$G_E(s) = C_s(sI - A_s)^{-1}B_s = \begin{bmatrix} (I + P_a(s))^{-1} - I & -\frac{1}{s}(I + P_a(s))^{-1} \\ (I + P_r(s))^{-1}sK(s)T_p(s) & (I + P_r(s))^{-1} - I \end{bmatrix} \tag{17}$$

Remark 1

Since the compensated system is asymptotically stable when there are no rate and amplitude saturation, the auxiliary variable $x_v(t)$ must satisfy the initial condition $x_v(0) = Hx_k(0)$.

3. STABILITY ANALYSIS OF THE COMPENSATED SYSTEM WITH RATE AND AMPLITUDE CONSTRAINTS

The global stability of the compensated system shown in Figure 2 is investigated in this section. From the discussion of last section, the compensated system with actuator saturation is globally stable if the equivalent system (13), (14) is asymptotically stable for all actuator limits.

Definition (Khalil [23])
A $p \times p$ proper rational transfer function matrix $Z(s)$ is said to be positive real if

(1) all elements of $Z(s)$ are analytic for $\mathrm{Re}[s] > 0$,
(2) any pure imaginary pole of any element of $Z(s)$ is a simple pole and the associated residue matrix of $Z(s)$ is positive semidefinite Hermitian, and,
(3) for all real ω for which $j\omega$ is not a pole of any element of $Z(s)$, the matrix $Z(j\omega) + Z'(-j\omega)$ is positive semidefinite.

The transfer function $Z(s)$ is strictly positive real (SPR) if $Z(s - \varepsilon)$ is positive real for some $\varepsilon > 0$, and is extended strictly positive real (ESPR) if $Z(s)$ is analytic in $\mathrm{Re}(s) \geqslant 0$ and satisfies $Z(j\omega) + Z'(-j\omega) > 0$ for $\omega \in [0, \infty]$ [23].

Remark 2
From the definitions of SPR, ESPR and Lemma 10.1 in Reference [23], if $Z(s)$ is SPR, then $Z(s)$ must be Hurwitz. If $Z(\infty) + Z'(\infty) > 0$, then $Z(s)$ is SPR if and only if $Z(s)$ is ESPR. It is shown that in Reference [24] ESPR transfer function matrix can be characterized by linear matrix inequality.

Definition (Khalil [23])
A memoryless nonlinearity $\psi:[0, \infty] \times R^p \to R^p$ is said to satisfy a sector condition if

$$[\psi(t, y) - K_{\min} y]^{\mathrm{T}} [\psi(t, y) - K_{\max} y] \leqslant 0, \quad \forall t \geqslant 0, \ \forall y \in \Gamma \subset R^p \tag{18}$$

for some real matrix $K_{\min}$ and $K_{\max}$, where $K = K_{\max} - K_{\min}$ is a positive definite symmetric matrix, and the interior of Γ is connected and contains the origin. If $\Gamma = R^p$, then $\psi(\cdot, \cdot)$ satisfies the sector condition globally, or simply, $\psi(\cdot, \cdot)$ belongs to the sector $[K_{\min}, K_{\max}]$. If (18) holds with strict inequality, then $\psi(\cdot, \cdot)$ is said to belong to the sector $(K_{\min}, K_{\max})$.

The following assumptions are necessary in deriving the global stability of the compensated system.

(A1): The plant and controller are all stable, i.e. the eigenvalues of A and F are in the open left-half plane.
(A2): The controller K is designed such that the closed-loop system without any actuator saturation is asymptotically stable.

The global stability of the compensated system is derived from the multivariable circle criterion [23], as given below.

Theorem 1

For the compensated system shown in Figure 2 satisfying assumptions (Al) and (A2), to be globally stable, the rate and amplitude saturation compensators (4) and (7) given by A_ξ, B_ξ, C_ξ, A_η, B_η and C_η are chosen such that (A_s, B_s) is controllable, (A_s, C_s) is observable and $Z_E(s) = I + G_E(s)$ is strictly positive real, where $G_E(s)$ is defined by (17).

From (13) , $G_E(s)$ has a state space realization:

$$G_E(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & 0 \end{array}\right]$$

It is clear that A_s is a stable matrix if $I + G_E(s)$ is SPR. For global stability, the nonlinearity $\varphi(v)$ belongs to the sector $[0, I]$, which satisfies the sector condition (18) globally with $K_{\min} = 0$ and $K_{\max} = I$. The system G_E is therefore globally stable from Lemma 10.1 in Reference [23] and thus $x_s(t)$ is convergent to the constant vector $-\alpha$ as $t \to \infty$. This also implies that the output of the compensated system is asymptotically stable, as $y = Cx_p$ and the global stability of the compensated system is established.

4. DESIGNING OF A RATE COMPENSATOR AND AN AMPLITUDE COMPENSATOR

In this section, a method to design the rate compensator $P_r(s)$ and the amplitude compensator $P_a(s)$ such that $I + G_E(s)$ to be SPR is given. Since the compensated system is globally stable with either the amplitude saturation compensator or the rate saturation compensation, the design of the rate and amplitude compensators is divided into two steps. Firstly, the amplitude saturation compensator $P_\eta(s)$ is designed such that the compensated system with no rate saturation is globally stable. Secondly, the rate saturation compensator $P_r(s)$ is chosen using the previous designed $P_\eta(s)$ such that $I + G_E(s)$ is SPR. The advantage of this design procedure is that the compensated system with both rate and amplitude saturation is always globally stable.

4.1. Design of the amplitude saturation compensator $P_\eta(s)$

From (13), the dynamics of the compensated system with only amplitude saturation compensation is given by

$$\dot{x}_a(t) = \begin{bmatrix} A & 0 & 0 & 0 \\ -GC & F & 0 & 0 \\ -\mathrm{HGC} & \mathrm{HF} & -D_\eta & C_\eta \\ 0 & 0 & -B_\eta & A_\eta \end{bmatrix} x_a(t) + \begin{bmatrix} -B \\ 0 \\ -D_\eta \\ -B_\eta \end{bmatrix} \bar{u}(t) \tag{19}$$

$$v(t) = [0 \;\; 0 \;\; I \;\; 0] x_a(t) \tag{20}$$

$$\bar{u}(t) = -u(t) = -\mathrm{sat}(v(t)) \tag{21}$$

where

$$x_a(t) = \left[x_p'(t) \;\; x_k'(t) \;\; x_v'(t) \;\; x_\eta'(t)\right]' + \beta$$

in which $\beta = [0\ \ (F^{-1}Gw)'\ \ 0\ \ 0]'$. The transfer function matrix of (19) and (20) is

$$T_a(s) = (I + P_a(s))^{-1}(I + K(s)T_p(s)) - I$$

From the circle criterion, the compensated system with only amplitude saturation is globally stable if $I + T_a(s)$ is SPR. The amplitude saturation compensator is designed by choosing A_η, B_η, C_η and D_η that satisfies this condition, which can be transformed to the following strictly positive control problem. Note that $I + T_a(s)$ can be expressed in terms of linear fractional transformation [25],

$$I + T_a(s) = \mathscr{F}_e(N_a(s), T_\eta(s))$$

where $N_a(s)$, $T_\eta(s)$ have the following state space realizations:

$$N_a(s) \stackrel{s}{=} \left[\begin{array}{c|cc} A_{a0} & B_{a1} & B_{a2} \\ \hline C_{a1} & I & 0 \\ C_{a2} & I & 0 \end{array}\right], \quad T_\eta(s) = D_\eta - P_\eta(s) \stackrel{s}{=} \left[\begin{array}{c|c} A_\eta & -B_\eta \\ \hline C_\eta & 0 \end{array}\right]$$

and

$$A_{a0} = \begin{bmatrix} A & 0 & 0 \\ -GC & F & 0 \\ -\mathrm{HGC} & \mathrm{HF} & -D_\eta \end{bmatrix}, \quad B_{a1} = \begin{bmatrix} -B \\ 0 \\ -D_\eta \end{bmatrix}$$

$$C_{a1} = C_{a2} = [0\ \ 0\ \ I], \quad B_{a2} = C'_{a1}$$

Clearly, the problem of finding $P_\eta(s)$ such that $I + T_a(s)$ is SPR can now be transformed to the strictly positive real control problem with an appropriate matrix D_η being chosen a prior. That is, it is equivalent to find a transfer function matrix $T_\eta(s)$ such that $\mathscr{F}_e(N_a(s), T_\eta(s))$ is SPR, which can be solved by the LMIs (see the remark following Theorem 4.1 of Reference [24]), as summarized in the following theorem.

Theorem 2

There exists a proper transfer matrix $P_\eta(s)$ such that $I + T_a(s)$ is SPR for some chosen matrix D_η if and only if there exist two positive definite matrices W_1, W_3 and matrices W_2, W_4 such that

(i)
$$\begin{bmatrix} A_{a0}W_1 + W_1A'_{a0} + B_{a2}W_2 + W'_2B'_{a2} & W_1C'_{a1} - B_{a1} \\ C_{a1}W_1 - B'_{a1} & -2I \end{bmatrix} < 0 \tag{22}$$

(ii)
$$\begin{bmatrix} A'_{a0}W_3 + W_3A_{a0} + W_4C_{a2} + C'_{a2}W'_4 & W_3B_{a1} + W_4 - C'_{a1} \\ B'_{a1}W_3 + W'_4 - C_{a1} & -2I \end{bmatrix} < 0 \tag{23}$$

(iii) The spectral radius $\rho(Y_0X_0) < 1$, where $X_0 = W_1^{-1}$ and $Y_0 = W_3^{-1}$. (24)

If (22)–(24) hold, then an amplitude compensator

$$P_a(s) = \frac{1}{s}P_\eta(s) = \frac{1}{s}(C_\eta(sI - A_\eta)^{-1}B_\eta + D_\eta)$$

can be obtained such that $I + T_a(s)$ is SPR. With the matrices A_η, B_η and C_η given by,

$$B_\eta = M_0 L_0 \quad C_\eta = F_0$$

$$A_\eta = A_{a0} + B_{a2} F_0 + M_0 L_0 C_{a2} + \Delta_0$$

where $F_0 = W_2 W_1^{-1}$, $L_0 = W_3^{-1} W_4$, $M_0 = (I - Y_0 X_0)^{-1}$ and

$$\Delta_0 = -\tfrac{1}{2}(B_{a1} + M_0 L_0)(C_{a1} - B'_{a1} X_0) + M_0 Y_0 R_0(X_0) - M_0 Y_0 F'_0 B'_{a2} X_0$$

$$R_0(X_0) = (A_{a0} + B_{a2} F_0)' X_0 + X_0 (A_{a0} + B_{a2} F_0) + \tfrac{1}{2}(C_{a1} - B'_{a1} X_0)'(C_{a1} - B'_{a1} X_0)$$

4.2. Design of the rate saturation compensator $P_r(s)$

Assume that $P_\eta(s)$ have now been designed as discussed in Section 4.1, such that the compensated system with only amplitude saturation is globally stable. From Theorem 1, the compensated system with rate and amplitude saturation is globally stable if $I + G_E(s)$ is SPR. Similarly, $I + G_E(s)$ can also be expressed as

$$I + G_E(s) = \mathscr{F}_e(N_r(s), T_r(s))$$

where $N_r(s)$, $T_r(s)$ have state space realizations, respectively:

$$N_r(s) \overset{s}{=} \left[\begin{array}{c|cc} A_{r0} & B_{r1} & B_{r2} \\ \hline C_{r1} & I & D_{r12} \\ C_{r2} & D_{r21} & 0 \end{array}\right], \quad T_r(s) \overset{s}{=} \left[\begin{array}{c|c} A_\xi - B_\xi C_\xi & -B_\xi \\ \hline C_\xi & 0 \end{array}\right]$$

and

$$A_{r0} = \begin{bmatrix} A & 0 & 0 & 0 \\ -GC & F & 0 & 0 \\ 0 & 0 & -D_\eta & C_\eta \\ 0 & 0 & -B_\eta & A_\eta \end{bmatrix}, \quad B_{r1} = \begin{bmatrix} -B & 0 \\ 0 & 0 \\ -D_\eta & -I \\ -B_\eta & 0 \end{bmatrix}, \quad B_{r2} = 0, \quad D_{r12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_{r1} = \begin{bmatrix} 0 & 0 & I & 0 \\ -\mathrm{HGC} & \mathrm{HF} & 0 & 0 \end{bmatrix}, \quad C_{r2} = [-\mathrm{HGC} \;\; \mathrm{HF} \;\; 0 \;\; 0], \quad D_{r21} = [0 \;\; I]$$

Therefore, the problem of choosing $P_r(s)$ such that $I + G_E(s)$ is SPR can also be transformed to a strictly positive real control problem, i.e. finding a transfer $T_r(s)$ such that $\mathscr{F}_e(N_r(s), T_r(s))$ is SPR. Similar to the design of the amplitude saturation compensator, this problem can be solved using the LMIs approach as given in Theorem 2. After $T_r(s)$ is selected, $P_r(s)$ is obtained by $P_r(s) = (I + T_r(s))^{-1} - I$.

Theorem 3

There exists a strictly proper transfer matrix $P_r(s)$ such that $I + G_E(s)$ is SPR if and only if there exist two positive definite matrices Q_1, Q_3 and matrices Q_2, Q_4 such that

(i)
$$\begin{bmatrix} A_{r0} Q_1 + Q_1 A'_{r0} & Q_1 C'_{r1} + Q'_2 D'_{r12} - B_{r1} \\ C_{r1} Q_1 + D_{r12} Q_2 - B'_{r1} & -2I \end{bmatrix} < 0 \qquad (25)$$

(ii)
$$\begin{bmatrix} A'_{r0}Q_3 + Q_3A_{r0} + Q_4C_{r2} + C'_{r2}Q'_4 & Q_3B_{r1} + Q_4D_{r21} - C'_{r1} \\ B'_{r1}Q_3 + D'_{r21}Q'_4 - C_{r1} & -2I \end{bmatrix} < 0 \tag{26}$$

(iii)
$$\text{The spectral radius } \rho(Y_1X_1) < 1, \text{ where } X_1 = Q_1^{-1} \text{ and } Y_1 = Q_3^{-1}. \tag{27}$$

If (25)–(27) hold, an amplitude saturation compensator $P_r(s) = C_\xi(sI - A_\xi)^{-1}B_\xi$ is obtained such that $I + G_E(s)$ is SPR, with the matrices A_ξ, B_ξ and C_ξ given by

$$B_\xi = M_1L_1, \quad C_\xi = F_1$$
$$A_\xi = B_\xi C_\xi + A_{r0} + M_1L_1C_{r2} + \Delta_2$$

where $F_1 = Q_2Q_1^{-1}$, $L_1 = Q_3^{-1}Q_4$, $M_1 = (I - Y_1X_1)^{-1}$ and

$$\Delta_2 = -\tfrac{1}{2}(B_{r1} + M_1L_1D_{r21})(C_{r1} - B'_{r1}X_1 + D_{r12}F_1) + M_1Y_LR_1(X_1) - \tfrac{1}{2}M_1Y_1F'_1D'_{r12}(C_{r1} - B'_{r1}X_1 + D_{r12}F_1)$$

$$R_1(X_1) = A'_{r0}X_1 + X_1A_{r0} + \tfrac{1}{2}(C_{r1} - B'_{r1}X_1 + D_{r12}F_1)'(C_{r1} - B'_{r1}X_1 + D_{r12}F_1)$$

Remark 3
For single-input single-output system, $P_\eta(s)$ can be obtained by another method. Since from Theorem 10.2 in Reference [23], the globally stable condition that $I + T_a(s)$ is SPR is equivalent to $\mathrm{Re}\{T_a(j\varpi)\} > -1$ as given in Reference [4]. Since a class of proper transfer functions $\bar{T}_a(s)$ that satisfies $\mathrm{Re}\{\bar{T}_a(j\varpi)\} > 1$ can always be found from the Nyquist plot of $\bar{T}_a(s)$ [4], $P_a(s)$ can be computed from the chosen $\bar{T}_a(s)$ as follows:

$$P_a(s) = (1 + K(s)T_P(s))(1 + \bar{T}_a(s))^{-1} - 1$$

where $K(s)$ and $T_P(s)$ are transfer matrices of the system P and controller K, respectively.

5. EXAMPLE

Consider the model of Bank-to-Turn Missile [26, 18],

$$\dot{x}_p(t) = Ax(t) + Bu(t) \quad y = cx$$

where

$$A = \begin{bmatrix} -0.818 & -0.999 & 0.349 \\ 80.29 & -0.579 & 0.009 \\ -2734 & 0.5621 & -2.10 \end{bmatrix}, \quad B = \begin{bmatrix} 0.147 & 0.012 \\ -194.4 & 37.61 \\ -2176 & -1093 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$x_p \triangleq [\text{sideslip, yawrate, rollrate}]', \quad u \triangleq [\text{rudder, aileron}]', \quad y \triangleq [\text{sideslip, yawrate}]'$$

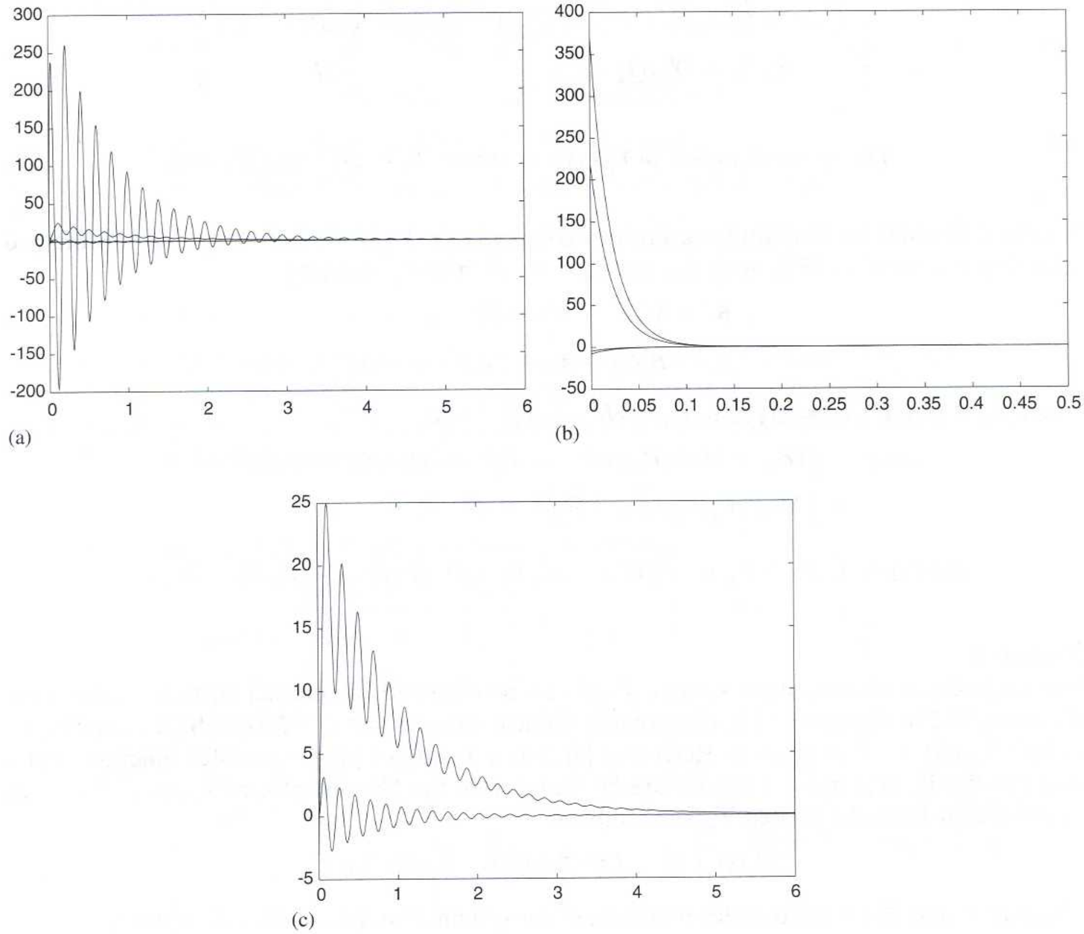

Figure 4. The closed-loop system without saturation: (a) System state $x(t)$; (b) $v(t)$ and $\dot{v}(t)$; and (c) system output $y(t)$.

H_∞ design methodology [27] is used to obtain a linear controller,

$$\dot{x}_k(t) = \begin{bmatrix} -47.1269 & 0 & 0 \\ 0 & -47.1269 & 0 \\ 0 & 0 & -47.1269 \end{bmatrix} x_k(t) + \begin{bmatrix} 0.009 & 0 \\ 0.002 & -0.06 \\ 0 & 0.002 \end{bmatrix} e(t)$$

$$v(t) = \begin{bmatrix} -0.5 & 1 & 0.7 \\ 0.3 & -1 & 8 \end{bmatrix} x_k(t)$$

$$e(t) = w(t) - y(t)$$

which ensures that the closed-loop system without rate and amplitude saturation is asymptotically stable and H_∞ norm of transfer function T_{yw} from w to y is less than 1. Let

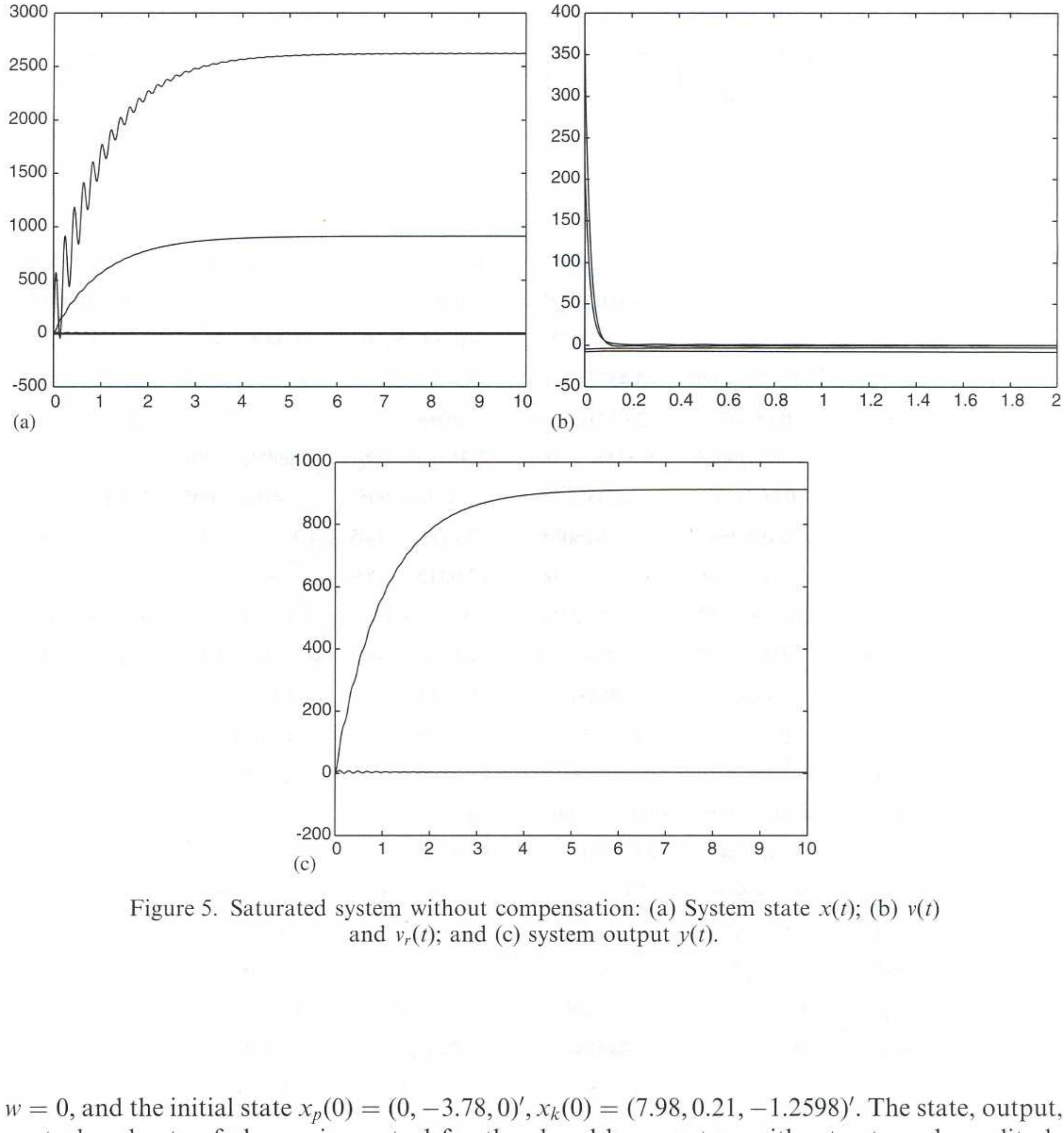

Figure 5. Saturated system without compensation: (a) System state $x(t)$; (b) $v(t)$ and $v_r(t)$; and (c) system output $y(t)$.

$w = 0$, and the initial state $x_p(0) = (0, -3.78, 0)'$, $x_k(0) = (7.98, 0.21, -1.2598)'$. The state, output, control and rate of change in control for the closed-loop system without rate and amplitude saturation are shown in Figure 4. The simulation is repeated with the actuator subject to the following rate and amplitude constraints: $u_1^+ = 8, u_1^- = -8$, and $\dot{u}_1^+ = -10$, $\dot{u}_1^- = -10$, let the initial state be $x_p(0) = (0, -3.78, 0)'$, $x_k(0) = (7.98, 0.21, -1.2598)'$, $x_v(0) = Hx_k(0) = (-4.6619, -7.8943)'$. The state, output, controller output and rate of controller output for the saturated system without compensation are shown in Figure 5. Note that $v(0) = (-4.6619, -7.8943)'$ and $v_r(0) = (219.4772, 372.3221)'$ giving rate saturation before amplitude saturation. From Figure 5, it is clear that the saturation system without rate and amplitude compensation is not asymptotically stable, as the output approaches a constant instead of zero for large t. Now, the rate and amplitude compensators are designed using the procedure presented in Section 4, as given below

$$A_\eta = \begin{bmatrix} -265 & 0 & 0 \\ 0 & -245 & 0 \\ 0 & 0 & -185 \end{bmatrix}, \quad B_\eta = \begin{bmatrix} 12 & 16 \\ 15 & -1 \\ -8.1 & -0.8 \end{bmatrix}$$

$$C_\eta = \begin{bmatrix} 1.8 & 4.8 & 14 \\ 8 & -5.6 & 0 \end{bmatrix}, \quad D_\eta = \begin{bmatrix} 295 & 0 \\ 0 & 855 \end{bmatrix}$$

$$A_\zeta = \left[\begin{matrix} -15.81 & -1.005 & 0.35115 & 1.7155e-006 & -6.9396e-007 & -1.2493e-005 \\ 80.535 & -15.471 & -0.024202 & -0.00027979 & 0.00065118 & -0.00029865 \\ -2750.2 & 0.31864 & -17.077 & -0.00025166 & 0.00067096 & -0.00084108 \\ -0.0090028 & 9.0862e-007 & -3.4575e-007 & -62.127 & -1.4256e-009 & 3.2223e-009 \\ -0.0014512 & 0.060076 & -2.0121e-005 & -2.2094e-007 & -62.127 & -2.8512e-007 \\ -1.8315e-005 & -0.0020025 & 6.6815e-007 & 7.3673e-009 & -1.6999e-008 & -62.127 \\ 1.1153 & 0.0044182 & 0.0053398 & 2.1413e-005 & -5.1416e-005 & 1.0816e-005 \\ 7.3646 & 0.066056 & -0.016085 & -2.1716e-005 & 4.3018e-005 & 4.4001e-005 \\ 0.19432 & 0.0010619 & 8.0476e-005 & -3.6042e-009 & 1.3879e-007 & -1.029e-006 \\ 0.056441 & 0.00085853 & 0.00012457 & 2.9672e-007 & -6.1249e-007 & 4.3734e-007 \\ -0.042194 & -0.00049025 & -6.5934e-005 & -1.1123e-007 & 2.2286e-007 & -6.1397e-008 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.5024 & -0.19905 & -0.055316 & -0.0016238 & -0.17028 \\ -165.54 & 20.213 & 0.13875 & -2.0907 & -6.4545 \\ -2206 & -1110.8 & -27.826 & -10.981 & -102.56 \\ -0.0001338 & 7.4588e-005 & 1.0014e-005 & -4.4992e-007 & 2.9168e-005 \\ 0.020709 & -0.010756 & 0.00021017 & 0.00030543 & 0.0013404 \\ -0.0006913 & 0.00035909 & -6.9317e-006 & -1.0184e-005 & -4.4465e-005 \\ -13.49 & 2.6466 & 2.904 & 7.8105 & 26.171 \\ 9.7757 & -15.62 & 21.956 & -15.624 & 2.0598 \\ 0.31872 & 0.062581 & -264.69 & -0.062599 & 0.54307 \\ 0.26581 & 0.035284 & 0.043738 & -244.82 & 0.65901 \\ -0.15001 & -0.023013 & -0.045453 & -0.078751 & -185.36 \end{matrix}\right]$$

$$B_\zeta = \left[\begin{matrix} -139.55 & -12061 & -7676.1 & 0.051153 & -7.6776 & 0.2563 \\ 175.36 & 15495 & 9678.8 & -0.065002 & 9.869 & -0.32944 \end{matrix}\right.$$

$$\left.\begin{matrix} 1442.6 & -1634.1 & 3.4253 & 4.1368 & -3.1039 \\ -1847.7 & 2121.9 & -4.2906 & -5.2555 & 3.9399 \end{matrix}\right]'$$

$$C_\zeta = \left[\begin{matrix} -0.0025318 & -0.0586 & -9.9505e-008 & -23.563 & 47.127 & 32.989 \\ 0.00067869 & 0.076 & -1.1995e-008 & 14.138 & -47.127 & 377.02 \end{matrix}\right.$$

$$\left.\begin{matrix} -0.00371 & -1.5044e-005 & -2.8534e-005 & -7.804e-005 & -0.0003143 \\ -1.5043e-005 & -0.0011946 & -3.8569e-005 & 2.8818e-005 & -1.7762e-006 \end{matrix}\right]$$

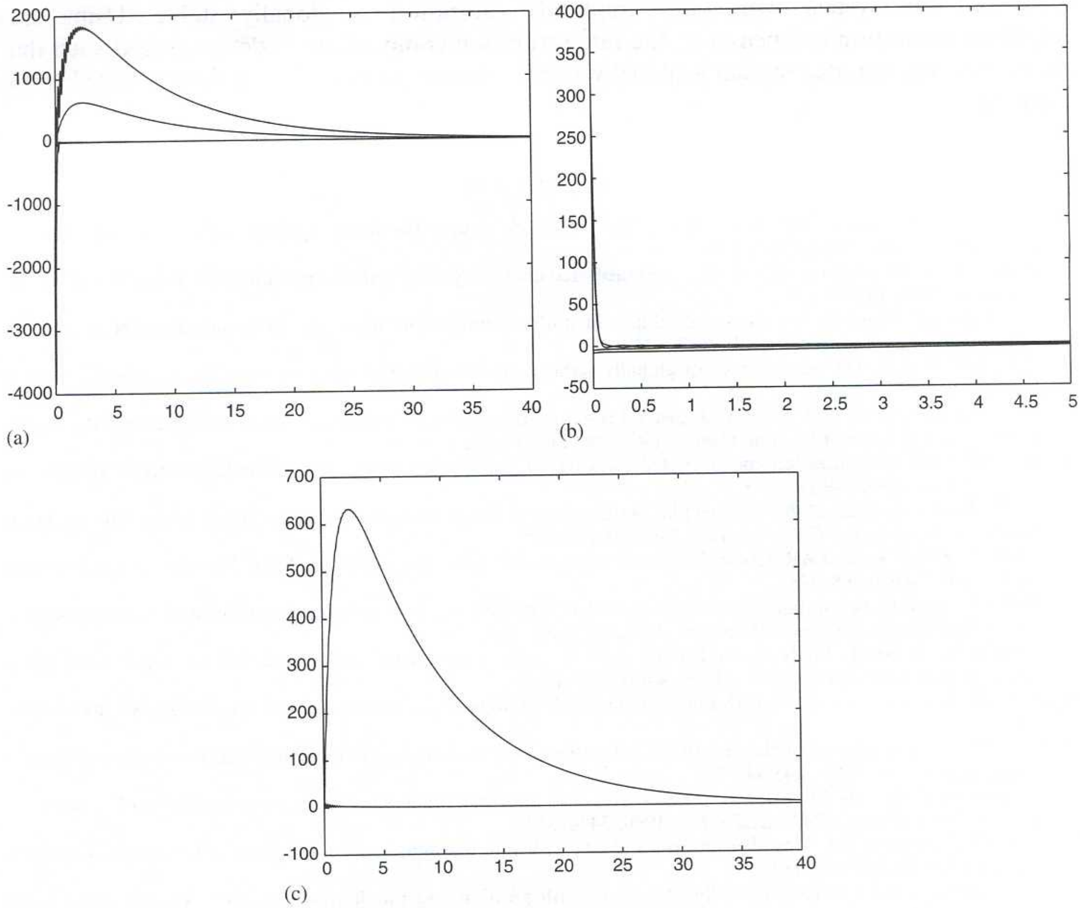

Figure 6. Saturated system with compensation: (a) System state $x(t)$; (b) $v(t)$ and $v_r(t)$; and (c) system output $y(t)$.

Let the initial state $x_s(0) = (0, -3.78, 0, 7.98, 0.21, -1.2598, -4.66185, -7.89432, 0_{1\times 14})'$, the state, output, control and rate of change in the control for the saturated system with both rate and amplitude compensators are shown in Figure 6. It is clear that the output of the compensated system is asymptotically stable, as it approaches to zero as t increases.

6. CONCLUSIONS

In this paper, the compensation for multivariable control systems subject to rate and amplitude saturation of the actuator is presented. It is assumed that the rate saturation occurs before the amplitude saturation, as this is more common in practical control systems. It is shown that the compensated system is globally stable if the equivalent system of the compensated system is strictly positive real. A procedure to design the rate and amplitude compensators based on this stability result is proposed. First, an amplitude saturation compensator is designed such that the system with only amplitude saturation is globally stable. Using this amplitude saturation compensator, the rate saturation compensator is designed next such that the overall compensated system is globally stable. The design procedure is demonstrated by an example.

REFERENCES

1. Sontag ED, Sussmann HJ. Nonlinear output feedback design for linear systems with saturating controls. *Proceedings of 29th IEEE CDC*, 1990; 3414–3416.
2. Fuller AT. In the large stability of relay and saturated control systems with linear controllers. *International Journal of Control* 1969; **10**:457–480.
3. Sussmann HJ, Yang Y. On the stabilizability of multiple integrators by means of bounded feedback controls. *Proceedings of 30th IEEE CDC*, 1991; 70–72.
4. Chan CW, Hui K. On the existence of globally stable actuator saturation compensators. *International Journal of Control* 1998; **69**:773–788.
5. Sussmann HJ, Sontag ED, Yang Y. A general result on the stabilization of linear systems using bounded controls. *IEEE Transactions on Automatic Control* 1994; **39**(12):2411–2425.
6. Teel AR. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems and Control Letters* 1992; **18**:165–171.
7. Lin Z, Saberi A. Semi-global exponential stabilization of linear systems subject to 'input saturation' via linear feedbacks. *Systems and Control Letters* 1993; **21**:225–239.
8. Saberi A, Zongli L, Teel AR. Control of linear systems with saturating actuators. *IEEE Transactions on Automatic Control* 1996; **41**(3):368–378.
9. Teel AR. Semi-global stabilization of linear null controllable systems with input nonlinearities. *Proceedings of American Control Conference*, Baltimore, MD, 1994; 947–951.
10. Cavallo A, De Maria G, Nistri P. Robust control design with integral action and limited rate control. *IEEE Transactions on Automatic Control* 1999; **44**(8):1569–1572.
11. Horowitz I. Feedback systems with rate and amplitude limiting. *International Journal of Control* 1984; **40**(6):1215–1229.
12. Chan CW, Hui K. Design of stable actuator saturation compensators in the frequency. *IEE Proceedings of Control Theory Applications* 1998; **145**:345–351.
13. Kapasouris P, Athans M. Control systems with rate and magnitude saturation for neutrally stable open loop systems. *Proceedings of 29th IEEE CDC*, 1990; 3404–3409.
14. Liao YK, Horowitz IM. Unstable uncertain plants with rate and amplitude saturations. *International Journal of Control* 1986; **44**(4):1147–1159.
15. Lin Z. Semi-global stabilization of linear systems with position and rate-limited actuators. *Systems and Control Letters* 1997; **30**:1–11.
16. Kapila V, Valluri S. Model predictive control of systems with actuator amplitude and rate saturation. *Proceedings of the 37th IEEE Conference on Decision and Control*, vol. 2. 1998; 1396–1401.
17. Angeli D, Mosca E, Casavola A. Output predictive control of linear plants with saturated and rate limited actuators. *Proceedings of American Control Conference* 2000; **1**(6):272–276.
18. Tyan F, Bernstein DS. Dynamic output feedback compensation for linear systems with independent amplitude and rate saturations. *International Journal of Control* 1997; **67**(1):89–116.
19. Hui K, Chan CW. A stability analysis of controllers subject to amplitude and rate constraints. *Proceedings of IEEE International Conference on Control Applications* 1997; 259–264.
20. Hui K, Chan CW. Control synthesis versus saturation compensation for systems with rate and amplitude constraints. *Proceedings of the American Control Conference*, Philadelphia, Pennsylvania, 1998.
21. Hui K. Design of rate and amplitude saturation compensators using three degrees of freedom controllers. *International Journal of Control* 2001; **74**(1):91–101.
22. Hui K, Chan CW. Noise attenuation of compensators for rate and amplitude constrained systems. *Automatica* 1999; **35**:973–979.
23. Khalil HK. *Nonlinear System*. Prentice-Hall: Upper Saddle River, New Jersey; 1996.
24. Weiqian SP, Khargonekar P, Shim D. Solution to the positive real control problem for linear time-invariant systems. *IEEE Transactions on Automatic Control* 1994; **39**(10):2034–2046.
25. Zhou K, Doyle JC. *Essentials of Robust Control*. Prentice-Hall: Upper Saddle River, New Jersey; 1998.
26. Rodriguez AA, Cloutier JR. Control of a Bank-to-Turn (BTT) missile with saturating actuators. *Proceedings of American Control Conference*, Baltimore, Maryland, June 1994; 1660–1664.
27. Iwasaki T, Skelton RE. All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica* 1994; **30**(8):1307–1317.

